Optimal Estimation of Large Toeplitz Covariance Matrices

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Outline

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- **Motivation from Asymptotic Equivalence Theory**
- **Main Results**
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Introduction

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be i.i.d. p-variate Gaussian with an unkown Toeplitz covariance matrix $\Sigma_{p\times p}$,

$$\begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{p-2} & \sigma_{p-1} \\ \sigma_1 & \sigma_0 & & & \sigma_{p-2} \\ \vdots & & \ddots & & \vdots \\ \sigma_{p-2} & & & \sigma_0 & \sigma_1 \\ \sigma_{p-1} & \sigma_{p-2} & \cdots & \sigma_1 & \sigma_0 \end{pmatrix}.$$

Goal: Estimate $\Sigma_{p\times p}$ based on the sample $\{\mathbf{X}_i : 1 \leq i \leq n\}$.

Introduction – Spectral Density Estimation

The model given by observing

$$\mathbf{X}_1 \sim N\left(0, \Sigma_{p\times p}\right)$$

with $\Sigma_{p\times p}$ Toeplitz is commonly called

Spectral Density Estimation

$\mathbf{X}_1$, a stationary centered Gaussian sequence with spectral density f

where

$$f(t) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\sigma_m \exp(imt) = \frac{1}{2\pi}[\sigma_0 + 2\sum_{m=1}\sigma_m \cos\left(mt\right)],\ t \in [-\pi, \pi].$$

Here we have $\sigma_{-m} = \sigma_m$.

Remark: there is a one-to-one correspondence between f and $\Sigma_{\infty\times\infty}$.

Introduction – Problem of Interest

We want to understand the minimax risk:

$$\inf_{\hat{\Sigma}} \sup_{\mathcal{F}} \mathbb{E}\|\hat{\Sigma} - \Sigma\|^2$$

where $\|\cdot\|$ denotes the spectral norm and $\mathcal{F}$ is some parameter space for f.

Motivation from Asymptotic Equivalence Theory

Golubev, Nussbaum and Z. (2010, AoS)

The **Spectral Density Estimation** given by observing each $\mathbf{X}_i$ is asymptotically equivalent to the **Gaussian white noise**

$$dy_i(t) = \log f(t)dt + 2\pi^{1/2}p^{-1/2}dW_i(t), \; t \in [-\pi, \pi]$$

under some assumptions on the unknown f.

For example,

$$\mathcal{F}_\alpha(M, \epsilon) = \left\{ f : |f(t_1) - f(t_2)| \le M |t_1 - t_2|^\alpha \text{ and } f(t) \ge \epsilon \right\}.$$

We need $\alpha > 1/2$ to establish the asymptotic equivalence.

Intuitively, the model

$$\mathbf{X}_i \sim N\left(0, \Sigma_{p\times p}\right), \; i = 1, 2, \ldots, n$$

is asymptotically equivalent to

$$dy\left(t\right) = \log f(t)dt + 2\pi^{1/2}\left(np\right)^{-1/2} dW\left(t\right), \; t \in [-\pi, \pi]$$

possibly under some strong assumptions on the unknown f .

“Equivalent” Losses

Let $\hat{\Sigma}_{\infty\times\infty}$ be a Toeplitz matrix and $\hat{f}$ be the corresponding spectral density.

We know

$$\left\|\hat{\Sigma}_{\infty\times\infty} - \Sigma_{\infty\times\infty}\right\| = 2\pi \left\|\hat{f} - f\right\|_{\infty}$$

based on a well known result

$$\|\Sigma_{\infty\times\infty}\| = 2\pi \|f\|_{\infty}$$

where

$$\|\Sigma_{\infty\times\infty}\| = \sup_{\|v\|_2=1} \|\Sigma_{\infty\times\infty} v\|_2 , \text{ and } \|f\|_{\infty} = \sup_x |f(x)| .$$

Intuitively

$$\left\|\hat{\Sigma}_{p\times p} - \Sigma_{p\times p}\right\| \approx \left\|\hat{\Sigma}_{\infty\times\infty} - \Sigma_{\infty\times\infty}\right\| ?$$

Thus optimal estimation on f may imply optimal estimation on Σ.

Question

Can we show

$$\inf_{\hat{\Sigma}_{p\times p}} \sup_{\mathcal{F}_\alpha} \mathbb{E} \left\| \hat{\Sigma}_{p\times p} - \Sigma_{p\times p} \right\|^2 \asymp \left(\frac{np}{\log(pn)} \right)^{-\frac{2\alpha}{2\alpha+1}} ?$$

Remark: Classical result on nonparametric function estimation under the sup norm:

$$\inf_{\hat{\Sigma}_{p\times p}} \sup_{\mathcal{F}_\alpha} \mathbb{E} \left\| \hat{f} - f \right\|_\infty^2 \asymp \left(\frac{np}{\log(pn)} \right)^{-\frac{2\alpha}{2\alpha+1}} .$$

Again,

- We don't really have the asymptotic equivalence.
- The following claim is very intuitive

$$\left\|\hat{\Sigma}_{p\times p} - \Sigma_{p\times p}\right\| \approx \left\|\hat{\Sigma}_{\infty\times\infty} - \Sigma_{\infty\times\infty}\right\|.$$

Main Results –Lower bound

Show that

$$\inf_{\hat{\Sigma}_{p\times p}} \sup_{\mathcal{F}_\alpha} \mathbb{E} \left\| \hat{\Sigma}_{p\times p} - \Sigma_{p\times p} \right\|^2 \geq c \left(\frac{np}{\log (pn)} \right)^{-\frac{2\alpha}{2\alpha+1}}$$

for some $c > 0$.

Main Results –Lower bound

A more informative model

Observe $\mathbf{Y}_1 = (\mathbf{X}_1, \mathbf{W}_1)$ with a circulant covariance matrix $\tilde{\Sigma}_{(2p-1)\times(2p-1)}$

$$\begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & \sigma_{p-2} & \sigma_{p-1} & \sigma_{p-2} & \cdots & \sigma_2 & \sigma_1 \\ \sigma_1 & \sigma_0 & & & \sigma_{p-2} & \sigma_{p-1} & & & \sigma_2 \\ \vdots & & \ddots & & \vdots & & \ddots & & \vdots \\ \sigma_{p-2} & & & \sigma_0 & \sigma_1 & & & \sigma_{p-1} & \sigma_{p-2} \\ \sigma_{p-1} & \sigma_{p-2} & \cdots & \sigma_1 & \sigma_0 & \sigma_2 & \cdots & \sigma_{p-2} & \sigma_{p-1} \\ & & & & \cdots\cdots\cdots\cdots & & & & \end{pmatrix}.$$

Define

$$\omega_j = \frac{2\pi j}{2p-1}, \; |j| \le p-1$$

and where

$$f_p(t) = \frac{1}{2\pi} \left(\sigma_0 + 2 \sum_{m=1}^{p-1} \sigma_m \cos(mt) \right).$$

It is well known that the spectral decomposition of $\tilde{\Sigma}_{(2p-1)\times(2p-1)}$ can be described as follows:

$$\tilde{\Sigma}_{(2p-1)\times(2p-1)} = \sum_{|j| \le p-1} \lambda_j \mathbf{u}_j \mathbf{u}_j'$$

where

$$\lambda_j = f_p(\omega_j), \; |j| \le p-1$$

and the eigenvector $\mathbf{u}_j$ doesn't depend on $\{\sigma_m : 0 \le m \le p-1\}$.

Main Results –Lower bound

The more informative model is *exactly* equivalent to

$$Z_j = f_p(\omega_j)\,\xi_j,\ \ |j| \leqslant p-1, Var(\xi_j) \asymp 1/n.$$

For this model it is easy to show

$$\sup_{\mathcal{F}_\alpha} \mathbb{E}\left\|\hat{f} - f\right\|_\infty^2 \geq c\left(\frac{np}{\log(pn)}\right)^{-\frac{2\alpha}{2\alpha+1}}.$$

Main Results –Lower bound

We have

$$\begin{aligned} \left\|\hat{\Sigma}_{p\times p}-\Sigma_{p\times p}\right\| &\geq \sup_{t\in[-\pi,\pi]}\left|(\sigma_0-\hat{\sigma}_0)+2\sum_{m=1}^{p}(1-\frac{m}{p})\left(\hat{\sigma}_m-\sigma_m\right)e^{imt}\right| \\ &= \sup_{t\in[-\pi,\pi]}\left|\hat{f}\left(t\right)-f\left(t\right)\right|+\text{negligible term} \end{aligned}$$

based on a fact

$$\|\Sigma_{p\times p}\|\geqslant \sup_{t\in[-\pi,\pi]}\frac{1}{p}\langle\Sigma_{p\times p}v_t,v_t\rangle=\sup_{t\in[-\pi,\pi]}\left|\sigma_0+2\sum_{m=1}^{p}(1-\frac{m}{p})\sigma_m e^{imt}\right|$$

where $v_t=(e^{it},e^{i2t},\cdots,e^{ipt})$. Thus

$$\sup_{\mathcal{F}_\alpha}\mathbb{E}\left\|\hat{\Sigma}_{p\times p}-\Sigma_{p\times p}\right\|^2\geq c\left(\frac{np}{\log\left(pn\right)}\right)^{-\frac{2\alpha}{2\alpha+1}}.$$

Remark: Need to have some assumptions on (n,p,α) such that the "negligible term" is truly negligible.

Main Results – Upper bound

Show that there is a $\hat{\Sigma}_{p\times p}$ such that

$$\sup_{\mathcal{F}_\alpha} \mathbb{E} \left\| \hat{\Sigma}_{p\times p} - \Sigma_{p\times p} \right\|^2 \leq C \left(\frac{np}{\log(pn)} \right)^{-\frac{2\alpha}{2\alpha+1}}$$

for some $C > 0$.

Main Results – Upper bound

Let $\Sigma_k = [\sigma_m 1_{\{m \leqslant k-1\}}]$ be a banding approximation of $\Sigma_{p\times p}$,, and $\tilde{\Sigma}_k$ be a banding approximation of the sample covariance matrix $\hat{\Sigma}_{p\times p}$. Note that $\mathbb{E}\tilde{\Sigma}_k = \Sigma_k$. Let $\hat{\Sigma}_k$ be a Toeplitz version of $\tilde{\Sigma}_k$ by taking the average of elements along the diagonal.

We have

$$\left\|\hat{\Sigma}_k - \Sigma_p\right\|^2 \leq 2\left\|\hat{\Sigma}_k - \Sigma_k\right\|^2 + 2\left\|\Sigma_k - \Sigma_p\right\|^2 \leq 8\pi^2\left(\| \hat{f}_k - f_k \|_\infty^2 + \| f_k - f_p \|_\infty^2\right)$$

since

$$\|\Sigma_k\| \leqslant 2\pi \, \| f_k \|_\infty = \sup_{[-\pi,\pi]} |\sigma_0 + 2\sum_{m=1}^{k-1} \sigma_m \cos(mt)|.$$

Main Results – Upper bound

Variance-bias trade-off

Variance part:

$$\mathbb{E} \parallel \hat{f}_k - f_k \parallel_\infty^2 \leq C \frac{k}{np} \log (np) .$$

Bias part:

$$\parallel f_k - f_p \parallel_\infty^2 \leq Ck^{-2\alpha}.$$

Set the optimal k : $k_{optimal} \asymp \left(\frac{np}{\log np}\right)^{\frac{1}{2\alpha+1}}$ which gives

$$\sup_{\mathcal{F}_\alpha} \mathbb{E} \left\| \hat{\Sigma}_{p\times p} - \Sigma_{p\times p} \right\|^2 \leq C \left(\frac{np}{\log (pn)} \right)^{-\frac{2\alpha}{2\alpha+1}}$$

Remark: For simplicity we consider only the case $k_{optimal} \leq p$.

Main Result

Theorem. The minimax risk of estimating the covariance matrix $\Sigma_{p\times p}$ over the class $\mathcal{F}_\alpha$ satisfies

$$\inf_{\hat{\Sigma}_{p\times p}} \sup_{\mathcal{F}_\alpha} \mathbb{E} \left\| \hat{\Sigma}_{p\times p} - \Sigma_{p\times p} \right\|^2 \asymp \left(\frac{np}{\log(pn)} \right)^{-\frac{2\alpha}{2\alpha+1}} ?$$

under some assumptions on (n, p, α).

Remarks

- Full asymptotic equivalence?
- Sharp asymptotic minimaxity?

Summary

- We studied rate-optimality of Toeplitz matrices estimation.
- Le Cam's theory plays important roles.
- Full asymptotic equivalence and sharp asymptotics remain unknown.